

# $U(1)_V \otimes U(1)_A$ symmetry breaking in superconductivity

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## Abstract

We argue that the general symmetry-breaking pattern in (quasi-)conventional (parity and time-reversal symmetric single-band spin-singlet) superconductivity is given by  $U(1)_V \otimes U(1)_A \rightarrow U(1)_A$ , where V stands for vector and A stands for axial-vector, as opposed to the breaking of  $U(1)_V \equiv U(1)_{\text{ele/mag}}$  by itself as is commonly thought. This symmetry-breaking pattern implies that there will be a Higgs mode which, together with the Goldstone boson that is absorbed by the photon (Meissner effect), characterize the symmetry-breaking dynamics. We obtain a number of strikingly simple analytical results, which amalgamate the findings of the standard BCS and Ginzburg–Landau theories.

**Keywords:** superconductivity

## 1. Introduction

We present a general analysis of the symmetry-breaking pattern found in (quasi-)conventional superconductivity (SC) [1, 2]. By (quasi-)conventional SC, we mean that the system can be considered to be invariant under parity (P) and time-reversal (T) transformations and consist of only one band of charge carriers, and that the SC pairs are spin-singlet. The pairing mechanism is not restricted to phonon exchange. Generalization is conceivable but will not be discussed in this study. In addition, we mainly consider s-wave SC. However, generalization will be discussed.

The charge carriers will be called ‘electrons’ and their charge conjugate will be called ‘holes’, with the understanding that the analysis applies equally to the reverse case.

We argue that the symmetry-breaking pattern is given by

$$U(1)_V \otimes U(1)_A \rightarrow U(1)_A, \quad (1)$$

where V stands for vector and A stands for axial-vector, as opposed to the breaking of  $U(1)_V \equiv U(1)_{\text{ele/mag}}$  by itself as is commonly thought<sup>1</sup>. By the Goldstone theorem, there will be one gapless excitation mode corresponding to the broken symmetry, i.e., the Goldstone boson, which is responsible for the Meissner effect, and one excitation mode, corresponding to the residual symmetry, with finite gap, i.e., the Higgs boson, which is an essential component of the symmetry breaking.

This identification allows us to evaluate the symmetry-breaking parameters analytically in an essentially non-perturbative (quasi-perturbative) way, by using the framework that has been developed by Gribov, myself and Das

[3, 4, 5]. In this framework, the parameters are calculated in terms of the dynamical degrees of freedom which, in the SC state, are the mixed states of electrons and holes.

The mixing of states and the resulting dynamical degrees of freedom will be discussed in sec. 2. The non-perturbative analysis will be carried out in secs. 3 onwards. The conclusions are stated at the end.

## 2. The fermionic degrees of freedom

Because the SC condensate (Cooper pair) is charged, charge conservation is violated in the electronic modes. That is, the system as a whole must conserve charge, but an electron can transform into a hole by emitting a Cooper pair. We thus have the following two two-point functions:

$$\begin{array}{c} e^{-i\phi_{SC}} \Delta_{SC}/2 \\ \longrightarrow \bullet \longleftarrow \end{array} \quad \text{and} \quad \begin{array}{c} e^{+i\phi_{SC}} \Delta_{SC}/2 \\ \longleftarrow \bullet \longrightarrow \end{array}, \quad (2)$$

with spin conservation in each case.  $\phi_{SC}$  is the phase of the SC condensate, and  $\Delta_{SC}$  is the absolute value of the superconducting gap. We may adopt  $\phi_{SC} = 0$  without loss of generality. As in the case of BCS theory [1, 2], the pairing interaction may have a finite energy range, and this restricts the applicability of eqn. (2) to be within that energy range. In this sense, and also in the sense of non-s-wave pairing symmetries,  $\Delta_{SC}$  is, in general,  $\mathbf{k}$ -dependent.  $\mathbf{k}$  is the wavevector. Note that the SC condensate itself is not a dynamical degree of freedom, even though the fluctuations in the SC condensate, i.e., the Goldstone and Higgs modes, are dynamical.

Let us apply eqn. (2) to the mixing of states using the Nambu representation [6]. We start from the case of

<sup>1</sup>Strictly speaking, the left-hand side of eqn. (1) should be divided by  $Z_2$  because of the existence of a non-trivial centre of the group.

ordinary metals (fig. 1). The dispersion relation can be denoted as:

$$(\psi_{\uparrow}^* \psi_{\downarrow}^*) \begin{pmatrix} \xi(\mathbf{k}) & 0 \\ 0 & \xi(\mathbf{k}) \end{pmatrix} \begin{pmatrix} \psi_{\uparrow} \\ \psi_{\downarrow} \end{pmatrix}. \quad (3)$$

Note that  $\xi(\mathbf{k}) \equiv \epsilon(\mathbf{k}) - \mu$ .

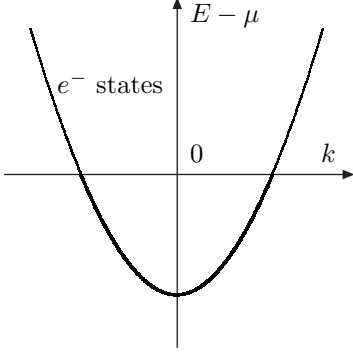


Figure 1: The dispersion relation in an ordinary metal (schematic). The zone boundaries are not shown. Thick line indicates occupied states.

We now take one of the two spin states, let us say  $\psi_{\downarrow}$ , and make a (CPT, where C stands for charge conjugation) conjugate of this state, such that:

$$\psi_{\downarrow} \longrightarrow \psi_{\downarrow}^* \equiv (\psi_{\downarrow})^{\text{CPT}} \equiv (\psi_{\downarrow})^* \equiv (\psi_{\uparrow}^*)_{\uparrow}. \quad (4)$$

That is, instead of considering the presence of a  $\psi_{\downarrow}$  state, we consider the absence of a hole state,  $(\psi^*)_{\uparrow}$ , which annihilates the  $\psi_{\downarrow}$  state. The dispersion relation of eqn. (3) is now replaced by the half-inverted dispersion relation:

$$(\psi_{\uparrow}^* \psi_{\downarrow}) \begin{pmatrix} \xi(\mathbf{k}) & 0 \\ 0 & -\xi(-\mathbf{k}) \end{pmatrix} \begin{pmatrix} \psi_{\uparrow} \\ \psi_{\downarrow}^* \end{pmatrix}. \quad (5)$$

This is depicted in fig. 2.

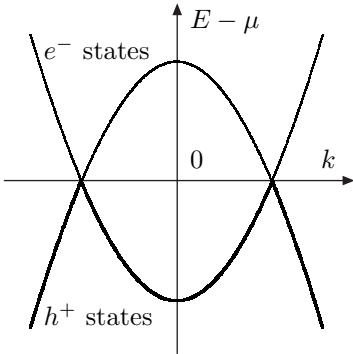


Figure 2: The half-inverted dispersion relation.

In the SC state, the presence of the two-point functions, as given by eqn. (2), implies that there will be off-diagonal terms in eqn. (5), that are given by

$$(\psi_{\uparrow}^* \psi_{\downarrow}) \begin{pmatrix} \xi(\mathbf{k}) & \Delta_{\text{SC}}/2 \\ \Delta_{\text{SC}}/2 & -\xi(-\mathbf{k}) \end{pmatrix} \begin{pmatrix} \psi_{\uparrow} \\ \psi_{\downarrow}^* \end{pmatrix}. \quad (6)$$

We have adopted  $\phi_{\text{SC}} = 0$

Let us make use of the parity invariance of the system, which implies  $\xi(-\mathbf{k}) = \xi(\mathbf{k})$ . We define the physical states  $\psi_{\text{u,d}}$  and mixing angle  $\theta_{\text{SC}}(\mathbf{k})$  (abbreviated as  $\theta$  in the following) by

$$\begin{pmatrix} \psi_{\text{u}} \\ \psi_{\text{d}} \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \psi_{\uparrow} \\ \psi_{\downarrow}^* \end{pmatrix}. \quad (7)$$

The diagonalization of eqn. (5) then yields

$$(\psi_{\text{u}}^* \psi_{\text{d}}^*) \begin{pmatrix} \tilde{\xi}(\mathbf{k}) & 0 \\ 0 & -\tilde{\xi}(\mathbf{k}) \end{pmatrix} \begin{pmatrix} \psi_{\text{u}} \\ \psi_{\text{d}} \end{pmatrix}, \quad (8)$$

where

$$\tilde{\xi}(\mathbf{k}) = \sqrt{\xi(\mathbf{k})^2 + \left(\frac{\Delta_{\text{SC}}}{2}\right)^2} \quad (9)$$

and the mixing angle is given by:

$$\tan 2\theta_{\text{SC}}(\mathbf{k}) = \frac{\Delta_{\text{SC}}}{2\xi(\mathbf{k})}. \quad (10)$$

This gives rise to the structure that is shown in fig. 3.

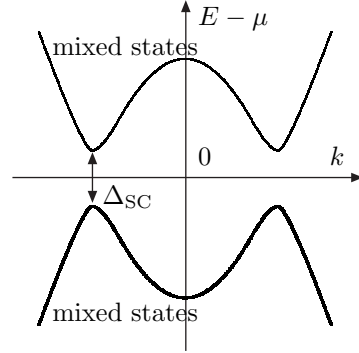


Figure 3: The dispersion relation in the SC state.

Note that the gap is given by  $\Delta_{\text{SC}}$ , as it should.

### 3. Symmetries of the system

Let us now focus on symmetries. Note that the usual electromagnetic U(1) symmetry, the U(1)<sub>V</sub> symmetry that is, is now associated with the rotation

$$\begin{pmatrix} \psi_{\uparrow} \\ \psi_{\downarrow}^* \end{pmatrix} \longrightarrow \exp(i\alpha_V \sigma_3) \begin{pmatrix} \psi_{\uparrow} \\ \psi_{\downarrow}^* \end{pmatrix}. \quad (11)$$

The presence of mixing implies that this is no longer a conserved symmetry. On the other hand, we can define another U(1) symmetry, i.e., the U(1)<sub>A</sub> symmetry, which is associated with the rotation

$$\begin{pmatrix} \psi_{\uparrow} \\ \psi_{\downarrow}^* \end{pmatrix} \longrightarrow \exp(i\alpha_A) \begin{pmatrix} \psi_{\uparrow} \\ \psi_{\downarrow}^* \end{pmatrix}. \quad (12)$$

This is conserved even with the mixing of the two states, because both states transform in the same way.

The  $U(1)_A$  group is an integral component of the symmetry-breaking pattern. To demonstrate this, consider the Goldstone boson that is associated with the breaking of the  $U(1)_V$  symmetry. We shall demonstrate later that its coupling to fermions is proportional to  $\sigma_2$ . However,  $\sigma_2$  is not invariant under  $U(1)_V$  rotation, and a linear combination of  $\sigma_1$  and  $\sigma_2$  arises, in general, under this rotation. There are therefore (at least) two bosonic modes that couple to fermions and, if one linear combination of  $\sigma_1$  and  $\sigma_2$  corresponds to the Goldstone boson, there must be the other linear combination, which corresponds to a mode with finite excitation energy, i.e., the Higgs boson. The existence of a non-elementary Higgs boson requires that there is a corresponding conserved symmetry which, in this case, is  $U(1)_A$ . In short, a closedness condition of the symmetry group implies that the symmetry is given by  $U(1)_V \otimes U(1)_A$ .

The  $U(1)_A$  transformation resembles the  $SU(2)_{\text{spin}}$  rotation around the  $z$  axis, but is not the same.  $U(1)_A$  rotation is the phase rotation that acts in the opposite direction after P conjugation. The above formulation, which adopts the Nambu representation, is merely a means. The same rotation can be defined by, for instance, dividing the phase space into two, and applying opposite phase rotation to each of the two sections.

Having said that, the above formulation makes it easy to make comparisons with magnetism in metals, especially ferromagnetism. Magnetism is where the energy level of a state is split according to spin by an exchange energy  $\Delta_{\text{ex}}$ . In our formulation, this is described by the dispersion relation

$$\begin{pmatrix} \psi_{\uparrow}^* & \psi_{\downarrow} \end{pmatrix} \begin{pmatrix} \xi(\mathbf{k}) - \frac{\Delta_{\text{ex}}}{2} & 0 \\ 0 & -\xi(-\mathbf{k}) - \frac{\Delta_{\text{ex}}}{2} \end{pmatrix} \begin{pmatrix} \psi_{\uparrow} \\ \psi_{\downarrow}^* \end{pmatrix}. \quad (13)$$

This is shown in fig. 4

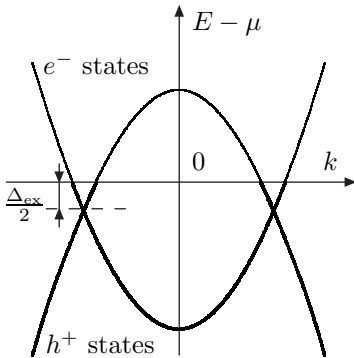


Figure 4: The dispersion relation for (ferro-)magnetic metals.

In the case of magnetism, the broken symmetry leads to the non-conservation of the spin current, which is restored by including the contribution of the Goldstone magnons.

The same should be the case in superconductivity, where the vector  $U(1)$  current becomes non-conserved.

Let us consider this in more detail. In the  $(\psi, \psi^*)$  basis, the non-conservation of current arises from the off-diagonal sector, which must be cancelled by the Goldstone-boson contribution. We first define the form factor  $f \equiv f_{\text{SC}}$  as the coupling strength of the current–Goldstone-boson two-point function where, in the four-component notation,

$$k_{\mu} \times \overset{\mu}{\bullet} - - - = i f D_G(k). \quad (14)$$

$k$  flows left to right.  $D_G$  is the Goldstone-boson propagator.  $f$  and  $D_G$  can be calculated by evaluating the two-point function, as we shall demonstrate later. Note that the Goldstone boson will be absorbed by the photon, which process being the cause of the Meissner effect.

The non-conservation of current and its restoration can be seen in various ways. One possibility, which we consider to be the simplest, is to consider the following Ward–Takahashi identity (in the  $(\psi, \psi^*)$  basis):

$$k_{\mu} \left[ \overset{\mu}{\bullet} - - - + \overset{\Gamma^{\mu}}{\curvearrowright} \overset{\Delta_{\text{SC}}}{\bullet} - - - + - - - \overset{\Delta_{\text{SC}}}{\bullet} \overset{\Gamma^{\mu}}{\curvearrowright} \right] = 0. \quad (15)$$

$\Gamma^{\mu}$  refers to the vector current vertex ( $\Gamma^0 = 1$ ), which by itself must satisfy the Ward–Takahashi identity, as usual. However, the sum of the second and the third terms does not satisfy the Ward–Takahashi identity, and the residual contribution must be cancelled by the first term. The same applies to the set of amplitudes in which the directions of the arrows are reversed. These two conditions fix the coupling of the Goldstone boson to fermions, which is found to have the form

$$f^{-1} \Delta_{\text{SC}} (\psi_{\uparrow}^* \psi_{\downarrow}) \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \psi_{\uparrow} \\ \psi_{\downarrow}^* \end{pmatrix} G. \quad (16)$$

The coupling in the  $(\psi_u, \psi_d)$  basis can be found by rotation. In particular, in the soft limit of the Goldstone boson, the coupling is completely off-diagonal. That is,  $\sigma_2$  is proportional to, and commutes with, the generator of the  $\theta_{\text{SC}}$  rotation.

The non-vanishing coupling of the Goldstone boson in the soft limit, which apparently is in contradiction with the so-called decoupling theorem, should not worry us, as the Ward–Takahashi identities take care of such pathological contributions. However, we need to introduce the Higgs degree of freedom to guarantee this. As explained before, this Higgs boson is associated with the residual  $U(1)_A$  symmetry.

The coupling of the Higgs to the fermions must have the same strength as the Goldstone boson, and must be of the same form as the  $\Delta_{\text{SC}}$  coupling. That is, the coupling is given by

$$f^{-1} \Delta_{\text{SC}} (\psi_{\uparrow}^* \psi_{\downarrow}) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \psi_{\uparrow} \\ \psi_{\downarrow}^* \end{pmatrix} (v + H). \quad (17)$$

One can use Ward–Takahashi identities to derive this more rigorously. Here,  $v$  is a constant, and carries the meaning of the vacuum expectation value of the SC condensate.  $f = 2v$  in order that this equation is consistent with eqn. (2). It follows that the vacuum expectation value of the SC condensate, and the penetration depth, can be calculated by finding the current–Goldstone-boson two-point function.

## 4. Two-point functions

There being only one Goldstone degree of freedom implies that the Goldstone boson must have a linear dispersion relation, for small  $\mathbf{k}$ . Let us parametrize  $D_G$  as:

$$D_G(E, \mathbf{k}) = \frac{1}{E^2 - u^2 \mathbf{k}^2 + i0}. \quad (18)$$

For this form of the Goldstone-boson Green's function, the Higgs-boson Green's function can be assumed to be given by:

$$D_H(E, \mathbf{k}) = \frac{1}{E^2 - u^2 \mathbf{k}^2 - \Delta_H^2 + i0}. \quad (19)$$

This is probably only an approximation, but this approximation makes it easy to impose current conservation. Specifically, we would like the following vertex to conserve the Ward–Takahashi identity:

$$\underline{G} - \overset{\mu}{*} - \underline{H} = \frac{\underline{G} - \overset{\mu}{\bullet} - \underline{H}}{-2i(k_G + k_H)^\mu} + \overset{\mu}{\bullet} \begin{array}{c} \text{---} \text{---} \text{---} \\ | \\ \underline{G} - \text{---} - \underline{H} \end{array} \quad (20)$$

The normalization of the vertex factor  $-2i(k_G + k_H)^\mu$  ( $u = 1$  for simplicity; all momenta flowing left to right) is determined by the requirement of the following Ward-Takahashi identity, again in  $(\psi, \psi^*)$  basis, which is a straightforward modification of eqn. (15):

$$k_\mu \left[ \text{diagram 1} + \text{diagram 2} + \text{diagram 3} \right] = 0. \quad (21)$$

Note that the form of the current is consistent with the form expected for the kinetic-energy term of the effective Lagrangian density:

$$\mathcal{L}_{\text{kin}} = \frac{1}{4} \left| \left( i\hbar \frac{\partial}{\partial t} - 2eA_0\sigma_3 \right) \Phi \right|^2 - \frac{u^2}{4} |(-i\hbar\nabla - 2e\mathbf{A}\sigma_3) \Phi|^2, \quad (2\mathfrak{A}) = 1/\sqrt{\mu_0 u^2 e^2 g_F}. \quad (30)$$

where  $\Phi = (v + H)\sigma_1 + G\sigma_2$ .

Using the Ward–Takahashi identity associated with eqn. (20), we then find that the  $GGH$  three-point coupling is given by:

$$g_{GGH} = 2f^{-1}\Delta_H^2 = \Delta_H^2/v. \quad (23)$$

Similarly to ref. [5], we find that this is consistent with the following form of the multi-bosonic Lagrangian density:

$$\mathcal{L}_{\text{bosons}} = -\frac{\Delta_H^2}{8v^2}(G^2 + (H+v)^2 - v^2)^2. \quad (24)$$

We shall not derive the other terms comprising this equation explicitly, since doing so will be merely repeating ref. [5].

As for the current–Goldstone-boson two-point function, a smarter method than to calculate this blindly is to calculate the current–current two-point function [7]. We shall omit the intermediate steps (see ref. [4]), but this gives us the following (one-loop) relation:

$$f^2 = -2 \int \frac{d^{d+1}k}{(2\pi)^{d+1}i} (\sin 2\theta_{\text{SC}}(\mathbf{k}))^2 G_{\text{u}}(k) G_{\text{d}}(k), \quad (25)$$

which is valid for small energy and momenta. We have neglected the contribution due to bosonic loops. We expect the bosonic loops to be suppressed because the Goldstone boson is absorbed by the photon. Note that the u states are empty and d states are occupied, so that:

$$f^2 = 2 \int \frac{d^d \mathbf{k} (\sin 2\theta_{\text{SC}}(\mathbf{k}))^2}{(2\pi)^d 2\sqrt{(\Delta_{\text{SC}}/2)^2 + \xi^2}}. \quad (26)$$

Let us replace the phase-space integration by:

$$2 \frac{d^d \mathbf{k}}{(2\pi)^d} = g(\mu + \xi) d\xi \approx g_F d\xi. \quad (27)$$

Here,  $g(\mu + \xi)$  refers to the density of states before symmetry breaking. We then obtain

$$f^2 \approx \frac{g_F}{2} \int_{-\infty}^{\infty} d(2\xi/\Delta_{\text{SC}}) [1 + (2\xi/\Delta_{\text{SC}})^2]^{-3/2}. \quad (28)$$

Strictly speaking, the limits of integration should be given by the relevant energy scale, e.g. the Debye frequency in the case of BCS superconductors. However, the integral is convergent and therefore we can take the limits to be  $\pm\infty$ . We then obtain

$$f^2 = 4v^2 \approx g_F. \quad (29)$$

That is, to the first approximation, the vacuum expectation value of the SC condensate is given solely by the electronic density of states in the normal state. This surprising conclusion implies that the penetration depth  $\lambda$ , for  $d = 3$ , is given by

$$2\lambda) = 1/\sqrt{\mu_0 u^2 e^2 g_F}. \quad (30)$$

This conclusion, as well as eqn. (29), is independent of the pairing symmetry. That is,  $\Delta_{\text{SC}}$  in the above equations can have an angular dependence without altering the results.

The effective Lagrangian terms for contributions other than the fermionic contributions (which also can be written down but decouple in the SC phase) are given by

$$\mathcal{L} = \mathcal{L}_{\text{bosons}} + \mathcal{L}_{\text{ele/mag}} + \mathcal{L}_{\text{kin}}. \quad (31)$$

The first term is given by eqn. (24). The second term is the usual Maxwellian term. The third term is given by eqn. (22).

By comparing against the corresponding Ginzburg–Landau terms, we then obtain eqn. (30) for the penetration depth, and

$$\xi = \frac{u\hbar\sqrt{2}}{\Delta_H}, \quad (32)$$

for the coherence length. The Ginzburg–Landau parameter  $\kappa$  is given by

$$\kappa = \lambda/\xi = \Delta_H/u^2\hbar\sqrt{2\mu_0e^2g_F}. \quad (33)$$

## 5. Tadpole cancellation and Higgs excitation energy

We now consider the tadpole cancellation condition:

$$- - \text{---} \bigcirc \text{---} + - - \text{---} \bigcirc \text{---} = 0. \quad (34)$$

The cancellation of the tadpole is a vacuum-stability condition. The second loop contains both Goldstone and Higgs bosons. This particular cancellation strategy is not the only possibility. Other mechanisms may be possible, depending on the nature of the pairing interaction. However, eqn. (34) should be understood as an all-order condition. This implies that the first loop includes all self-energy type corrections. Hence the only other possibility for cancelling the tadpole is, in our opinion, the mixing of the Higgs boson with some other bosonic mode that has the same quantum numbers as the vacuum (see ref. [4]).

The fermionic loop of eqn. (34) yields

$$- \int \frac{d^{d+1}k}{(2\pi)^{d+1}} f^{-1} \Delta_{\text{SC}} (-\sin 2\theta_{\text{SC}}) G_d(k). \quad (35)$$

This reduces to

$$f^{-1} g_F \int d\xi \frac{(\Delta_{\text{SC}}/2)^2}{\sqrt{(\Delta_{\text{SC}}/2)^2 + \xi^2}}. \quad (36)$$

This integral is divergent, and therefore needs to be cut-off. Cutting off the integral at  $\omega_{\text{cut}}$ , we obtain

$$\frac{g_F \Delta_{\text{SC}}^2}{2f} \sinh^{-1} \frac{2\omega_{\text{cut}}}{\Delta_{\text{SC}}} \approx \frac{g_F \Delta_{\text{SC}}^2}{2f} \ln \frac{\omega_{\text{cut}}}{\Delta_{\text{SC}}}. \quad (37)$$

As for the bosonic loop, let us say that the Goldstone-boson loop is suppressed because of the absorption of the Goldstone boson in the photon. For the Higgs-boson contribution, we obtain:

$$\frac{1}{2} \int \frac{d^{d+1}k}{(2\pi)^{d+1}} 6f^{-1} \Delta_H^2 D_H(k). \quad (38)$$

Using eqn. (19), we then obtain

$$\int \frac{-(3f^{-1} \Delta_H^2) d^d \mathbf{k}}{(2\pi)^d 2\sqrt{u^2 \mathbf{k}^2 + \Delta_H^2}}. \quad (39)$$

It is possible to show, by considering the spatial component of the vector current–current two-point function, that  $u = \mathcal{O}(v_F)$ , where  $v_F$  is the Fermi velocity in the normal state. Therefore  $u^2 \mathbf{k}^2 \gg \Delta_H^2$  near the zone boundaries where  $|\mathbf{k}| \approx K \approx \pi/a$ . In this limit, the integral is evaluated to be

$$\begin{cases} -3\Delta_H^2 K/4\pi u f & (d=2), \\ -3\Delta_H^2 K^2/8\pi^2 u f & (d=3). \end{cases} \quad (40)$$

Next, let us calculate the Higgs-boson excitation energy. The relevant Feynman graphs are shown in fig. 5.

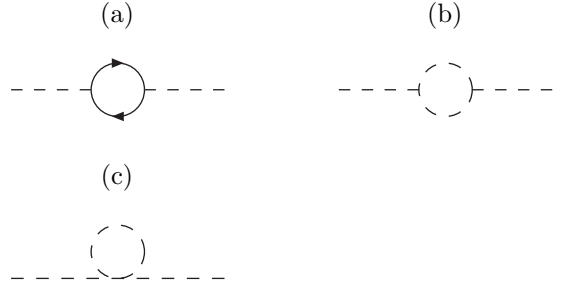


Figure 5: The diagrams for the self-energy of the Higgs (and Goldstone) boson. Diagram b involves  $GGH$  and  $HHH$  vertexes, and diagram c involves  $GGHH$  and  $HHHH$  vertexes.

Let us calculate  $\Delta_H$  as the sum of the self-energy contributions at zero external momentum. For the contribution of fig. 5a, we obtain

$$2 \int \frac{d^{d+1}k}{(2\pi)^{d+1}} (f^{-1} \Delta_{\text{SC}} \cos 2\theta_{\text{SC}})^2 G_u(k) G_d(k). \quad (41)$$

This reduces to

$$- \frac{g_F}{2} \int d\xi \frac{(f^{-1} \Delta_{\text{SC}} \cos 2\theta_{\text{SC}})^2}{\sqrt{(\Delta_{\text{SC}}/2)^2 + \xi^2}}. \quad (42)$$

This is negative and logarithmically divergent.

Here we find an analogy to our previous analyses [4, 5] in that such divergent contributions are cancelled by tadpoles. Corresponding to fig. 5c, we have the contribution

$$- \frac{1}{2} \int \frac{d^{d+1}k}{(2\pi)^{d+1}} 12f^{-2} \Delta_H^2 D_H(k). \quad (43)$$

This is  $-2f^{-1}$  times the bosonic tadpole that is given by eqn. (38). This implies that this contribution is equal to  $+2f^{-1}$  times the fermionic tadpole contribution that is given by eqn. (36). As a result, the net contribution to the Higgs-boson self-energy is given by

$$\frac{g_F}{2} \int d\xi \frac{(f^{-1} \Delta_{\text{SC}} \sin 2\theta_{\text{SC}})^2}{\sqrt{(\Delta_{\text{SC}}/2)^2 + \xi^2}}. \quad (44)$$

This is positive and finite, and is in fact equal to eqn. (26) times  $f^{-2} \Delta_{\text{SC}}^2$ . This implies that

$$\Delta_H = \Delta_{\text{SC}}. \quad (45)$$

As for the contribution of fig. 5b, this cannot be large. We may write down the amplitude as

$$-\frac{1}{2} \int \frac{d^{d+1}k}{(2\pi)^{d+1}i} (6f^{-2}\Delta_H^2)^2 (D_H(k))^2, \quad (46)$$

but using

$$(D_H(k))^2 = \frac{\partial}{\partial(\Delta_H^2)} D_H(k), \quad (47)$$

we find that the integral vanishes in the limit  $uK \gg \Delta_H$ . Having said that, eqn. (45) is not protected by any symmetry, and so it is possible that this degeneracy of  $\Delta_H$  and  $\Delta_{SC}$  is lifted by corrections particularly near the SC gap threshold.

Since eqns. (37) and (39) must cancel when added together, we now obtain

$$\Delta_{SC} \approx \omega_{\text{cut}} \exp \left( -\frac{3}{ug_F} \left( \frac{K}{2\pi} \right)^{d-1} \right). \quad (48)$$

This is identical to the BCS result [2], but with the phonon coupling  $\Lambda$  replaced by

$$\Lambda \longrightarrow \frac{u}{3} \left( \frac{2\pi}{K} \right)^{d-1}. \quad (49)$$

Let us return to the discussion of the coherence length. From eqn. (32) and using  $\Delta_H \approx \Delta_{SC}$ , we now obtain

$$\xi \approx \frac{u\hbar\sqrt{2}}{\Delta_{SC}}. \quad (50)$$

That is, the coherence length is inversely correlated with the SC gap.  $u \sim v_F$  is usually not expected to vary drastically. From this equation and eqn. (30), we can eliminate  $u$  which is difficult to calculate accurately, and obtain

$$\lambda\xi\Delta_{SC}\sqrt{g_F} \approx \sqrt{\frac{2\hbar^2}{\mu_0 e^2}}. \quad (51)$$

Other than  $g_F$ , which refers to the normal-state density-of-states, all parameters on the left-hand side characterize the SC state. We have carried out some preliminary estimations based on this equation, and found that there is indeed an inverse correlation between  $\lambda\xi$  and the SC critical temperature  $T_c$ , both of which are well measured. Further work is needed to test eqn. (51).

## 6. Conclusions

We argued that the symmetry-breaking pattern in superconductivity is given by  $U(1)_V \otimes U(1)_A \rightarrow U(1)_A$ , as opposed to the breaking of  $U(1)_V \equiv U(1)_{\text{ele/mag}}$  by itself as is commonly thought.

This allows us to calculate the parameters of the symmetry-breaking in an essentially non-perturbative way, and we obtained a number of strikingly simple results which need to be compared with experiment insofar as this is possible. These are:

- The presence of a collective excitation mode (Higgs), which occurs with a gap that is close to  $\Delta_{SC}$ .
- An expression for the BCS phonon coupling, eqn. (49).
- The result that the vacuum expectation value of the SC condensate is determined almost solely by the normal-state density-of-states, and simple expressions for the penetration depth and coherence length. In particular, eqn. (51) needs to be tested.

A natural extension of this work will be to the case of the co-existence of magnetism and SC, especially in the context of high- $T_c$  superconductivity [8]. In this regard, we expect that the anti-ferromagnetic ratio rule [4] will be satisfied in the SC nodal directions, i.e., the density-of-states curve is concave. An experimental verification of the same will be a useful initial step towards the analysis of this problem.

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